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CHARACTERISTICS OF DISCRETE PROPAGATION MODES
ON A SYSTEM OF HORIZONTAL WIRES OVER A
DISSIPATIVE EARTH

Department of Electrical Engineering
University of Colorado

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An equation for determining the propagation constants of the discrete modes of propagation on a system of m parallel thin wires above a dissipative earth is derived. Approximations to the Sommerfeld integrals involved in the modal equation that are accurate for relatively large values of the earth's refractive index are found and rigorous errors bounds for these approximations are obtained. Plots of the complex propagation constants for a dual bare wire line are calculated. A degeneracy is shown to

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exist between the two bifilar modes and in addition, one of the bifilar modes is seen to disappear into the improper Riemann sheet for small spacings between the two wires.

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CHARACTERISTICS OF DISCRETE PROPAGATION MODES

ON A SYSTEM OF HORIZONTAL WIRES

OVER A DISSIPATIVE EARTH

Steven W. Plate, David C. Chang and Edward F. Kuester

I. Introduction

Wave propagation along a system of horizontal wires parallel to the
earth's surface is of interest in many areas: the transmission properties
of overhead power lines, the performance of antenna arrays above ground,
the use of Goubau-lines in continuous ground wave detection systems,and
so on. At present, most investigations of this problem have been limited
to a low-frequency approximation similar to that first used for the single
wire case by Carson (see the bibliography in [1]). These analyses and
various refinements to them [2-14] are not sufficient to describe higher
frequency characteristics of these wires. Only very recently have more
rigorous analyses of the problem appeared: in 1959, Kostenko [15] analyzed
the case of three identical conductors at the same height; in 1963, Perel'man
[16] analyzed the general m-wire problem. A specialization of the latter
analysis to the two-wire line was done [17], as well as an independent solu-
tion of the m-wire problem [18], both in 1965. Most recently, Olsen and
Chang [19] have investigated the types of modes which can exist on a two-
wire line over a conducting earth in light of the earlier discovery of the
existence of so-called "earth-attached" modes on a single-wire system [1,20].

In this report, the equation for determining the propagation constants
of the discrete modes of propagation on a system of m parallel lines above
a dissipative earth is derived. Our basic formulation of the problem follows

closely that of Perel'man [16]. More specifically, we assume under the
thin-wire approximation, that the current on each conductor is uniformly
distributed around the surface of the conductor and totally in the axial
direction. A modal equation in the form of a determinant is then derived
by imposing an impedance boundary condition in which we assume that the
average axial electric field equals the axial surface impedance multi-
plied by the total axial current on the conductor. Obviously, this approxi-
mation is good only if the conductor radius is small compared to the freespace
wavelength, the distances between the individual conductors, and the heights
of the conductors above the earth.

We find approximations to the Sommerfeld integrals involved in the
modal equation that are accurate for relatively large values of the earth's
refractive index. We then use this approximate modal equation to calcu-
late the propagation constants for a dual line of bare wires. We find that
there are two monofilar modes and one or two bifilar modes depending on the
spacing between the wires. We also find that a modal degeneracy exists be-
tween the two bifilar modes for a particular set of parameters.

II. The Modal Equation

Consider a system of m infinitely long parallel thin wires located
over a plane interface between two half spaces of electrical constants
ϵ_1, μ_1 and ϵ_2, μ_2, σ_2 respectively. The j-th wire has a radius of a_j
(meters), and is located at a height $x=h_j$ (meters), and a position $y=d_j$
(meters). Figure 1 illustrates the geometry of the problem. The current
on the j-th wire is assumed to be of the form $I_j \exp(ik_1 \alpha z - i\omega t)$ where
α is the yet undetermined propagation constant relative to that of region
1, ω is the angular frequency of the fields, and $k_1 = \omega(\mu_1 \epsilon_1)^{\frac{1}{2}}$.
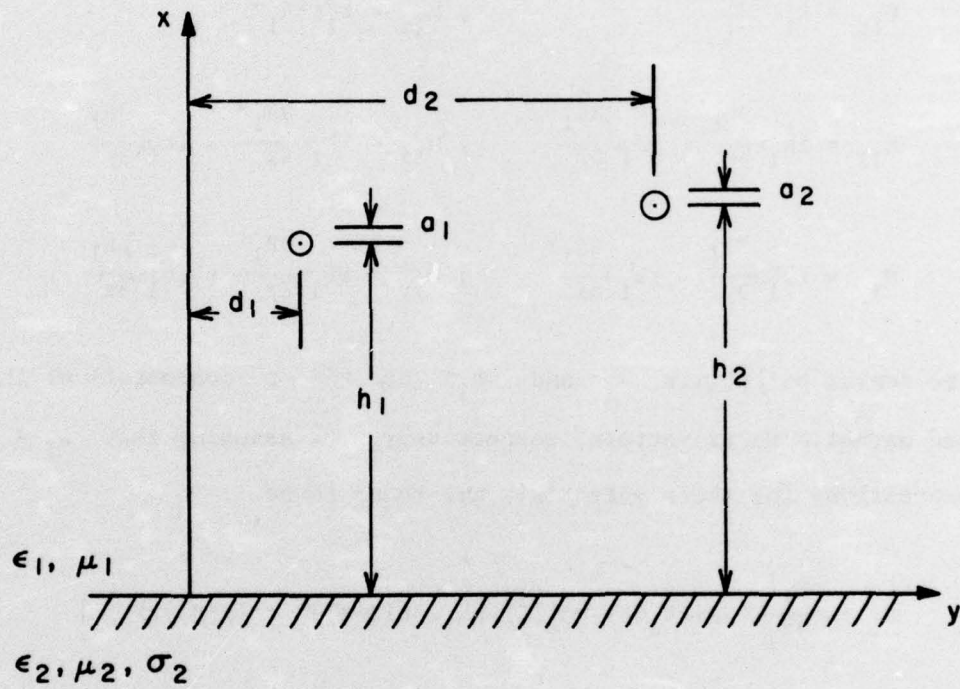
Fig. 1: Geometry for special case of two-wire line
over the earth.

Following Wait [21], expressions for the electric and magnetic fields in the region $x>0$ due to a filament of current I located at $x=x'$, $y=y'$ can be written as

$$E_{1z} = k_1^2 \zeta^2 \Pi_1 \qquad\qquad ; \; H_{1z} = k_1^2 \zeta^2 \Pi_1{}^*$$

$$E_{1x} = ik_1\alpha\frac{\partial \Pi_1}{\partial x} + i\mu_1\omega\frac{\partial \Pi_1{}^*}{\partial y} \qquad ; \; H_{1x} = ik_1\alpha\frac{\partial \Pi_1{}^*}{\partial x} - i\epsilon_1\omega\frac{\partial \Pi_1}{\partial y}$$

$$E_{1y} = ik_1\alpha\frac{\partial \Pi_1}{\partial y} - i\mu_1\omega\frac{\partial \Pi_1{}^*}{\partial x} \qquad ; \; H_{1y} = ik_1\alpha\frac{\partial \Pi_1{}^*}{\partial y} + i\epsilon_1\omega\frac{\partial \Pi_1}{\partial x} \; . \qquad (1)$$

The scalar potentials Π_1 and $\Pi_1{}^*$ are the z components of the electric and magnetic Hertz vectors, respectively. By assuming that $\mu_1 = \mu_2 = \mu_0$ expressions for these potentials are found to be

$$\Pi_1 = \frac{-\eta_1 I}{4k_1} \exp(ik_1\alpha z) \{ H_0^{(1)} [k_1\zeta ((x-x')^2 + (y-y')^2)^{\frac{1}{2}}]$$

$$- H_0^{(1)} [k_1\zeta ((x + x')^2 + (y - y')^2)^{\frac{1}{2}}] +$$

$$\frac{2}{i\pi\zeta^2} \int_{-\infty}^{\infty} [\frac{1}{u_1 + u_2} - \frac{\alpha^2}{u_2 + n^2 u_1}] \exp[-u_1 k_1(x + x') - i\lambda k_1(y - y')] \; d\lambda \} \quad (2)$$

$$\Pi_1{}^* = \frac{I\alpha(n^2 - 1)}{2\pi k_1 \zeta^2} \exp(ik_1\alpha z) \int_{-\infty}^{\infty} \frac{\lambda \exp[-u_1 k_1(x+x') - i\lambda k_1(y-y')]}{(u_2 + n^2 u_1)(u_1 + u_2)} \; d\lambda \qquad (3)$$

where

$$u_1 = (\lambda^2 - \zeta^2)^{\frac{1}{2}} \qquad ; \; -\pi/2 \le \arg u_1 < \pi/2$$

$$u_2 = (\lambda^2 - \zeta_n^2)^{\frac{1}{2}} \qquad ; \; -\pi/2 \le \arg u_2 < \pi/2$$

- 5-

$$\zeta = (1 - \alpha^2)^{\frac{1}{2}} \qquad ; \ 0 \le \arg \zeta < \pi$$

$$\zeta_n = (n^2 - \alpha^2)^{\frac{1}{2}} \qquad ; \ 0 \le \arg \zeta_n < \pi$$

The arguments of these variables are defined so that the electromagnetic fields are bounded at infinity. In these expressions $H_0^{(1)}(X)$ represents the Hankel function of the first kind and of order zero, and n represents the relative refractive index of region 2 compared to region 1, defined as $n^2 = k_2^2 / k_1^2 = \epsilon_2(1 + i\delta)/\epsilon_1$ with δ (loss tangent) given as $\sigma_2/(\omega\epsilon_2)$ and $0 \le \arg n < \pi/4$. Finally, $\eta_1 = (\mu_0/\epsilon_1)^{\frac{1}{2}}$ is the intrinsic impedance of region 1. We note that the first and second terms in (2) are the direct contributions due to the current filament and its image in region 2, respectively. The integral in (2) and the magnetic potential given in (3) are due to the finite conductivity of region 2.

With the use of the addition theorem of cylindrical functions, we therefore obtain the expression for the electric potential due to a uniform current distributed on the surface of the j-th wire as

$$\Pi_{1j} = \frac{-\eta_1 I_j}{4\zeta^2 k_1} \exp(ik_1\alpha z) J_o(\zeta a_j k_1) \{\zeta^2 H_o^{(1)}[\zeta k_1((x-h_j)^2 + (y - d_j)^2)^{\frac{1}{2}}]$$

$$-\zeta^2 H_0^{(1)}[\zeta k_1((x + h_j)^2 + (y - d_j)^2)^{\frac{1}{2}}] + P[\alpha; k_1(x + h_j), k_1(y - d_j)]$$

$$-Q[\alpha; k_1(x + h_j), k_1(y - d_j)]\} \tag{4}$$

where

$$P(\alpha; X, Y) = \frac{2}{i\pi} \int_{-\infty}^{\infty} \frac{\exp(-Xu_1 - iY\lambda)}{u_1 + u_2} d\lambda \tag{5}$$

$$Q(\alpha; X, Y) = \frac{2\alpha^2}{i\pi} \int_{-\infty}^{\infty} \frac{\exp(-Xu_1 - iY\lambda)}{u_2 + n^2 u_1} d\lambda \tag{6}$$

and $X = k_1 x$ and $Y = k_1 y$, are the normalized distances in the x,y directions. If now we denote the average axial electric field on the k-th wire due to the current on the j-th wire as E_{kj}, we obtain from (1) and (4), together with the addition theorem of Bessel functions, the following expression:

$$E_{kj} = \frac{-\eta_1 k_1 I_j}{4} \exp(ik_1 \alpha z) J_0(\zeta A_j) J_0(\zeta A_k) \{\zeta^2 H_0^{(1)}[\zeta((H_j - H_k)^2 + (D_j - D_k)^2)^{\frac{1}{2}}]$$

$$- \zeta^2 H_0^{(1)}[\zeta((H_j + H_k)^2 + (D_j - D_k)^2)^{\frac{1}{2}}] + P(\alpha; H_j + H_k, D_k - D_j)$$

$$- Q(\alpha; H_j + H_k, D_k - D_j)\} \tag{7}$$

where $A_j = k_1 a_j$, $H_j = k_1 h_j$, and $D_j = k_1 d_j$. It is worth noting that if $j = k$, then (7) should be modified by replacing the first Hankel function term in the curly bracket by $H_0^{(1)}(\zeta A_k)/J_0(\zeta A_k)$ since the axial electric field E_{kk} is uniformly distributed in this case.

Thus, within the framework of a thin-wire approximation, a modal equation is found by setting the average axial electric field on the k-th wire to be equal to the axial surface impedance times the total axial current on the wire, i.e.,

$$\sum_{j=1}^{m} E_{kj}(\alpha) = Z_k(\alpha) I_k \qquad \text{for } k=1,2,\ldots,m \tag{8}$$

If now we define

$$M_{kj} = -4[E_{kj}/I_j - Z_j \delta_{ki}]/(k_1 \eta_1) \tag{9}$$

so that [M] is a $m \times m$ matrix with elements M_{kj}, we have from (8) and (9) the following matrix equation

$$[M][I] = 0 \tag{10}$$

where [I] is the $m \times 1$ column matrix with elements I_j. Thus in order for this to have a non-trivial solution (i.e. some $I_j \neq 0$), the determinant of [M] must be zero, therefore the resulting modal equation is

$$M(\alpha) = \det [M] = 0 \tag{11}$$

III. PROPAGATION MODES ON A DUAL-LINE

In the special case of two identical bare wires of equal height $(H_1 = H_2)$, the modal equation can be factored to obtain two solutions: the monofilar one where $I_1 = I_2$, and the bifilar one where $I_1 = -I_2$. The two independent modal equations are

$$0 = M_{\pm}(\alpha) = J_0^2(\zeta A)\{\zeta^2[H_0^{(1)}(\zeta A)/J_0(\zeta A) - H_0^{(1)}(\zeta 2H)] + P(\alpha;2H,0) - Q(\alpha;2H,0)\}$$

$$\pm J_0^2(\zeta A)\{\zeta^2[H_0^{(1)}(\zeta D) - H_0^{(1)}(\zeta R)] + P(\alpha;2H,D) - Q(\alpha;2H,D)\} \qquad (12)$$

where

$$A = A_1 = A_2, \quad H = H_1 = H_2, \quad D = |D_1 - D_2|, \quad R = [(2H)^2 + D^2]^{\frac{1}{2}}$$

We note that the Bessel function $J_0(\zeta A)$ is only important in the behavior of $M(\alpha)$ for large ζ. For the purpose of finding the roots of $M(\alpha)$, the term $J_0(\zeta A)$ may be approximated by unity under the thin wire assumption that $A \ll 1$.

The method we adopt in this report to find the zeros of $M_{\pm}(\alpha)$ is an iterative scheme similar to the well-known Newton's method. To use this method the derivative of $M(\alpha)$ with respect to α is needed. This is given by

$$M_{\pm}'(\alpha) = \{-2\alpha[H_0^{(1)}(\zeta A) - H_0^{(1)}(\zeta 2H)] + \alpha\zeta[AH_1^{(1)}(\zeta A) - 2H_1^{(1)}(\zeta 2H)]$$

$$+ P'(\alpha;2H,0) - Q'(\alpha;2H,0)$$

$$\pm\{-2\alpha[H_0^{(1)}(\zeta D) - H_0^{(1)}(\zeta R)] + \alpha\zeta[DH_1^{(1)}(\zeta D) - RH_1^{(1)}[\zeta R)]$$

$$+ P'(\alpha;2H,D) - Q'(\alpha;2H,D)\} \qquad (13)$$

where

$$P'(\alpha;X,Y) = \frac{-2\alpha}{i\pi} \int_{-\infty}^{\infty} \frac{1 + Xu_2}{u_1 u_2 (u_1 + u_2)} \exp(-u_1 X - i\lambda Y)d\lambda \tag{14}$$

$$Q'(\alpha;X,Y) = \frac{2\alpha}{i\pi} \int_{-\infty}^{\infty} \left[2u_1 - \alpha^2 X - \frac{\alpha^2(u_1 + n^2 u_2)}{u_2(u_2 + n^2 u_1)} \right] \frac{\exp(-u_1 X - i\lambda Y}{u_1(u_2 + n^2 u_1)} \, d\lambda \tag{15}$$

The $(j+1)$-th approximation to the root α_{j+1} is then obtained from the previous approximation α_j using the equation

$$\alpha_{j+1} = \alpha_j - M_\pm(\alpha_j)/M'_\pm(\alpha_j), \qquad (j \geq 0) \tag{16}$$

where α_0 is an initial guess to the root, and the iteration is stopped when $|M(\alpha_i)| < \epsilon$ (where ϵ is the desired accuracy of the root). Obviously the iteration converges only if the initial guess is close enough to the actual root. The method used in this report to locate the roots is to find one for a particular set of parameters (by trial and error or from previously known results and then vary the parameters by small increments, finding the root at each step until the desired set of parameters is reached. At each step the initial guess α_0 is the root of the previous set of parameters.

The integrals P and Q and their derivatives as given in (5), (6), (14), and (15), in principle may be evaluated numerically, however it is more efficient to find approximate expressions initially in order to avoid excessive computation in the complex α-plane. In the following section, we shall discuss various approximations valid for different ranges of parameters involved.

IV. APPROXIMATIONS FOR THE INTEGRALS
$P(\alpha;X,Y)$ and $Q(\alpha;X,Y)$

Using methods similar to those employed by Olsen and Chang [20], the integrals P and Q can be approximated in terms of known functions for the common case where the wire height above the ground is greater than the skin-depth of the ground, that is, $|n|\,H \gg 1$. The integral P as given in (5) can be written as

$$P(\alpha;X,Y) = -\frac{2}{i\pi N^2}\int_{\infty}^{\infty}(u_1-u_2)\exp(-u_1X - i\lambda Y)d\lambda \qquad (17)$$

where

$$N = (n^2 - 1)^{\frac{1}{2}} ; \quad 0 \le \arg N \le \pi/4$$

In most cases, we expect the useful solutions to the modal equation $M_{\pm}(\alpha) = 0$ to be located near $\alpha = 1$. This means that ζ is small so the integrand of (17) decays as $\exp(-X|\lambda|)$ away from the point $\lambda = 0$. The major contribution to the integral is then from the region $\lambda \approx 0$. We therefore expand u_2 in a Taylor series about $\lambda = 0$ to obtain

$$u_2 = -i\zeta_n + \frac{i\lambda^2}{2\zeta_n} + \frac{i\lambda^4}{8\zeta_n^3} + \cdots . \qquad (18)$$

This series converges only when $|\lambda| < |\zeta_n|$, however the integrand of (17) will have decayed by the factor $\exp(-X|\zeta_n|)$ outside these limits. Therefore an approximation based on the first few terms of (18) appears to be valid for $|\zeta_n|^2 \gg |\zeta|^2$ and $|\zeta_n|X \gg 1$. By using the first term of (18), we may rewrite (17) as

$$P(\alpha;X,Y) = \frac{2}{i\pi N^2}\int_{\infty}^{\infty}[u_1 + i\zeta_n]\exp(-u_1X - i\lambda Y)d\lambda$$

$$+ \frac{2}{i\pi N^2}\int_{\infty}^{\infty}[-i\zeta_n - u_2]\exp(-i_1X - i\lambda Y)d\lambda \qquad (19)$$

$$= P_0(\alpha;X,Y) + \varepsilon_{P0}$$

Neglecting ε_{P0}, we obtain a first order approximation to P as

$$P(\alpha;X,Y) \approx P_0(\alpha;X,Y)$$

$$= \frac{2}{i\pi N^2} \{[\partial^2/\partial X^2 - i\zeta_n \partial/\partial X] \int_{-\infty}^{\infty} [\exp(-u_1 X - i\lambda Y)/u_1]d\lambda$$

$$= (2/N^2)\{\zeta H_1^{(1)}(\zeta R)[i\zeta_n X/R + (X^2 - Y^2)/R^3]$$

$$- (\zeta^2 X^2/R^2)H_0^{(1)}(\zeta R) \tag{20}$$

where $R = (X^2 + Y^2)^{\frac{1}{2}}$. In deriving (19) the following identity has been used.

$$H_0^{(1)}(\zeta R) = (i\pi)^{-1} \int_{\infty}^{\infty} [\exp(-u_1 X - i\lambda Y)/u_1]d\lambda \tag{21}$$

An upper bound for the absolute error in this approximation is found in Appendix A to be

$$|\varepsilon_{P0}| < \frac{4[2 + 2\delta X + \delta^2 X^2 + \delta^3 X^3/3]}{\pi|N^2\zeta_n|X^3} \tag{22}$$

where

$$\delta = \begin{cases} 0 & (\text{Re } \zeta^2 \leq 0) \ . \\ (\text{Re } \zeta^2)^{\frac{1}{2}} & (\text{Re } \zeta^2 > 0) \ . \end{cases}$$

From this expression it is apparent that the error is small if $|n^3|X^3 \gg 8/\pi$ and if $|n^3| \gg 4\delta^3/(3\pi)$. For example say $n = 5.3 + i.95$ and Re $\zeta^2 < .04$, then the error for $h = .1\lambda$ is less than 9×10^{-3} and for $h = .5\lambda$ the error is less than 7×10^{-5}. These error estimates are conservative and as will be shown in a later section, the results obtained using this approximation are generally much better than what is indicated by the error bound. For the interested reader, we have also included in Appendix A higher order approximations and their error estimates.

Following a similar procedure, we now rearrange the integral Q as

$$Q(\alpha;X,Y) = \frac{2\alpha^2 n^2}{i\pi(n^4-1)} \int_\infty^\infty \frac{u_1 - u_2/n^2}{\lambda^2 - \lambda_p^2} \exp(-u_1 X - i\lambda Y)d\lambda \qquad (23)$$

where

$$\lambda_p = (\zeta^2 - 1/\hat{n}^2)^{\frac{1}{2}} \ ; \qquad 0 \le \arg \lambda_p < \pi$$

$$\hat{n} = (n^2+1)^{\frac{1}{2}} \ ; \qquad 0 \le \arg \hat{n} < \pi/4 \ .$$

Since the integrand of (23) has a pair of poles at $\pm\lambda_p$, it is more convenient to expand u_2 around $\lambda = \lambda_p$ instead of around $\lambda = 0$, so that

$$u_2 = \frac{-in^2}{\hat{n}} + \frac{i(\lambda^2 - \lambda_p^2)\hat{n}}{2n^2} + \frac{i(\lambda^2 - \lambda_p^2)^2\hat{n}^3}{8n^6} + \cdots \ . \qquad (24)$$

Based upon (24), we may rewrite (23) as

$$Q(\alpha;X,Y) = \frac{2\alpha^2 n^2}{i\pi(n^4-1)} \{ \int_\infty^\infty \frac{u_1 + i/\hat{n}}{\lambda^2 - \lambda_p^2} \exp(-u_1 X - i\lambda Y)d\lambda + \int_{-\infty}^\infty \frac{-i/\hat{n} - u_2/n^2}{\lambda^2 - \lambda_p^2} \exp(-u_1 X - i\lambda Y)d\lambda \}$$

$$= Q_o(\alpha;X,Y) + \varepsilon_{Qo} \qquad (25)$$

where

$$Q_o(\alpha;X,Y) = \frac{2\alpha^2 n^2}{i\pi(n^4-1)} \int_\infty^\infty \frac{\exp(-u_1 X - i\lambda Y)}{u_1 - i/\hat{n}} \ d\lambda$$

and

$$\varepsilon_{Qo} = -\frac{2\alpha^2}{i\pi(n^4-1)} \int_\infty^\infty \frac{\exp(-u_1 X - i\lambda Y)}{u_2 - in^2/\hat{n}} \ d\lambda$$

We note that the integrand of Q_o contains a pair of poles at $\lambda = \pm \lambda_p$, whereas the integrand of ε_{Qo} does not. According to Olsen and Chang [20] this pole will cause a singularity in Q where $\lambda_p = 0$ (i.e. at $\alpha^2 = 1 - 1/\hat{n}^2$). Since ε_{Qo} is small compared to Q_o and has a nonsingular integrand, we may neglect it to obtain the first order approximation

$$Q(\alpha;X,Y) \approx Q_0(\alpha;X,Y)$$

$$= \frac{2\alpha^2 n^2}{i\pi(n^4-1)} \left\{ \int_\infty^\infty [\exp(-u_1 X - i\lambda Y)/u_1] d\lambda + i/\hat{n} \int_{-\infty}^\infty \frac{\exp(-u_1 \dot{X} - i\lambda Y)}{u_1(u_1 - i/\hat{n})} d\lambda \right\}$$

$$= \frac{2\alpha^2 n^2}{(n^4-1)} H_0^{(1)}(\zeta R) + \frac{2\alpha^2 n^2}{\pi(n^4-1)\hat{n}} W(\alpha;X,Y) \qquad (26a)$$

where $W(\alpha;X,Y)$ can be written as

$$W(\alpha;X,Y) = \exp(-iX/\hat{n}) \left\{ \int_{-\infty}^\infty \frac{[\exp(-X(u_1 - i/\hat{n}))-1]}{u_1(u_1-i/\hat{n})} \exp(-i\lambda Y) d\lambda \right.$$

$$\left. + \int_\infty^\infty \frac{\exp(-i\lambda Y)}{u_1(u_1-i/\hat{n})} d\lambda \right\}$$

$$= \exp(-iX/\hat{n})\{W_X(\alpha;X,Y) + W_0(\alpha;Y)\} \qquad (26b)$$

We note that the integrand of W_X has a pair of branch cuts in the complex λ-plane due to the definition of u_1 but does not have any poles. According to Olsen and Chang [20] this results in a pair of branch cuts in the α-plane due to the motion of the branch points of u_1 (at $\lambda = \pm \zeta$) crossing the real axis in the λ plane. The branch points in the α-plane are at $\alpha = \pm 1$ and the cut is defined by those points where ζ is a real number.

The integrand of W_0 has, in addition, two poles at $\lambda = \pm\lambda_p$. Again we know from the work of Olsen and Chang [20] that these poles can cause branch cuts in the α-plane whenever they cross the real axis in the λ-plane because of the discontinuity in the residue calculation at $\pm\lambda_p$. These branch cuts in the α-plane are located at $\alpha = \pm (1 - S^2 - 1/\hat{n}^2)^{\frac{1}{2}}$ where S is any real number.

An expression for $W_0(\alpha;Y)$ is derived in Appendix B. One form of the expression is

$$W_0(\alpha;Y) = (2/\lambda_p)\cos(\lambda_p Y)[\ell n(\zeta) - \ell n(1/\hat{n} - i\lambda_p)] - (\pi/\lambda_p)\sin(\lambda_p Y)$$

$$+ \pi/(\lambda_p \hat{n}) \int_0^Y \sin[\lambda_p(Y-s)]H_0^{(1)}(\zeta s)\,ds \qquad (27)$$

where the principal values of the logarithms are chosen. A series expansion for the finite integral used above is given as

$$(iY/2) \sum_{m=0}^{\infty} \frac{(i\lambda_p Y)^m}{m!} [\exp(-i\lambda_p Y) - (-1)^m \exp(i\lambda_p Y)]I_m(\zeta Y) \qquad (28)$$

where $I_m(\zeta Y)$ is expressed in terms of known functions in equations (B.14), (B.15), and (B.16).

An expression for $W_x(\alpha;X,Y)$ is found in Appendix C to be

$$W_x(\alpha;X,Y) = -i\pi \int_0^X \exp(is/\hat{n})H_0^{(1)}[\zeta(s^2+Y^2)^{\frac{1}{2}}]ds \qquad (29)$$

A series representation of this integral can be shown to be

$$W_x(\alpha;X,Y) = \sum_{m=0}^{\infty} \frac{f_m(\zeta,X^2/2 + Y^2)X^{2m+1}}{m!} I_m(iX/\hat{n}) \qquad (30)$$

where f_m and I_m are again expressed in terms of known functions in Appendix C. We note that such a series does not converge well when Y small compared to X. In order to find an expression that is good in this region, $W(\alpha;X,Y)$ can be rearranged as

$$W(\alpha;X,Y) = \int_{-\infty}^{\infty} \frac{\exp(-u_1 X)\cos(\lambda Y)}{u_1(u_1 - i/\hat{n})}\,d\lambda \qquad (31)$$

An expression for (31) is found in Appendix D by expanding the cosine term into a Taylor series, and integrating this series term by term. The result is repeated here as

$$W(\alpha;X,Y) = \cos(\lambda_p Y)W(\alpha;X,0) - i\pi \sum_{m=1}^{\infty} \frac{(-1)^m Y^{2m}}{(2m)!} R_m(\alpha;X) \qquad (32)$$

where

$$W(\alpha;X,0) = \exp(-iX/\hat{n}) \{ -i\pi \int_0^{\overset{\chi}{}} \exp(is/\hat{n}) H_0^{(1)}(\zeta s) ds$$

$$+ (2/\lambda_p)[\ln(\zeta) - \ln(1/\hat{n} - i\lambda_p)]\}.$$

Explicit expressions for $W(\alpha;X,0)$ can be found in Appendix B and will not be repeated here. The terms R_m are expressed in terms of known functions in (D.4), and a series expansion for the finite integral is given as

$$\sum_{m=0}^{\infty} \frac{(iX/\hat{n})^m \; \gamma}{m!} I_m(\zeta X) \qquad (33)$$

where $I_m(\zeta X)$ is again given in (B.14), (B.15), and (B.16). It should be noted that the series in (32) converges only if $Y < X$, so in computing W we use (32) if $X^2 > 2Y^2$, otherwise we use (27), (28), and (30) inserted into (26b).

In the preceding approximations on the integral Q, it is assumed that in summing each of the series enough terms can be included to obtain any desired accuracy. As evident from (25), the term ε_{Qo} is omitted in the derivation. An upper bound for the error due to neglecting ε_{Qo} is found in Appendix E to be

$$|\varepsilon_{Qo}| < \frac{4}{\pi} \left| \frac{\alpha^2 n}{(n^4-1)n^2} \right| \frac{(1 + \delta X)}{X} \qquad (34)$$

where

$$\delta = \begin{cases} 0 & (\text{Re } \zeta^2 \le 0) \;, \\ (\text{Re } \zeta^2)^{\frac{1}{2}} & (\text{Re } \zeta^2 > 0) \;. \end{cases}$$

From this expression it is apparent that the error is small if $|n^5|X \gg 4/\pi$ and $|n^5| \gg 4\delta/\pi$. For example say $n = 5.3 + i.95$ and Real $\delta^2 < .04$, then the error for $h = 0.1\lambda$ is less than 3×10^{-4} and for $h = 0.5\lambda$ the error is less than 6×10^{-5}. We also note that Q is of the order $1/n^2$, so the relative error is of the order $1/n^3$. This error estimate is again conservative and the results we obtain in the following section will show that the error is usually less than indicated by these estimates. Similar to the evaluation of $P(\alpha)$ we have also included in Appendix E higher order approximations and their error estimates. It should be noted the series for P and Q, generated by including the higher order terms of u_2 are only asymptotic, so in general one does not necessarily improve the accuracy by including more terms.

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V. NUMERICAL RESULTS

We have developed a computer program to compute the roots of the dual-
line modal equation (12) using the first order approximations to the Sommerfeld
integrals $P(\alpha;X,Y)$ and $Q(\alpha;X,Y)$. To test the accuracy of these approxi-
mations we compared the values of the propagation constants found using the
first order approximations, to the values found using a numerical integration
of the Sommerfeld integrals. Typical results given in Table 1 show that the
accuracy of the approximations is quite good (on the order of 10^{-4} or less),
even for the case of a poorly conducting earth with $|n| \approx 5.4$ considered
here. It is assumed that subsequent values obtained using the approximate
modal equation are of the same order of accuracy as the values in Table 1.

In Fig. 2 the roots of the modal equation are plotted for several
values of the wire spacing d, with the wire height $h = 0.2\lambda$, radius
$a = .005\lambda$, and refractive index $n = 5.3 + i0.95$. As expected for large
spacings there are monofilar and bifilar modes with propagation constants
close to the values of the single wire modes. As the spacing is decreased
the attenuation of the "quasi TEM" monofilar mode increased until the spacing
is approximately equal to the wire height. After this point the propagation
constant approaches that of a single wire of radius equal to the geometric
mean of a and d. This can be shown directly from the modal equation.
The "earth-attached" monofilar mode is relatively insensitive to the spacing
of the wires. The "quasi TEM" bifilar mode is affected less by the earth
as the spacing decreases, because the fields are concentrated between the
two wires. It should be noted that the "earth-attached" monofilar mode
is less attenuated than the "quasi TEM" bifilar mode for spacings larger than

TABLE 1

$n = 5.3 + i0.95$ $a = .005\lambda$ $d = .2\lambda$

	α_{NUM}	α_{APPROX}	Error
$h = .4\lambda$			
α_{m1}	$0.9919776 + i1.4673 \times 10^{-2}$	$0.9919776 + i1.4661 \times 10^{-2}$	1.2×10^{-5}
α_{m2}	$0.9955308 + i9.4423 \times 10^{-4}$	$0.9955297 + i9.6029 \times 10^{-4}$	1.6×10^{-5}
α_{B1}	$0.9999414 + i5.2261 \times 10^{-4}$	$0.9999439 + i5.2627 \times 10^{-4}$	4.4×10^{-6}
$h = .15\lambda$			
α_{m1}	$0.9975878 + i4.0203 \times 10^{-2}$	$0.9977231 + i4.0272 \times 10^{-2}$	1.5×10^{-4}
α_{m2}	$0.9903529 + i1.8962 \times 10^{-3}$	$0.9903263 + i1.9349 \times 10^{-3}$	4.7×10^{-5}
α_{B1}	$1.0017878 + i7.7008 \times 10^{-3}$	$1.0019770 + i7.9703 \times 10^{-3}$	3.3×10^{-4}

Comparison of the propagation constants computed using the approximations to the
Sommerfeld integrals, to the values found using a numerical integration of the integrals
for the discrete modes of the dual conducting wires above the earth. α_{m1}, α_{m2} and α_{B1}
are the propagation constants for the two monofilar and the bifilar modes respectively.
n is the refraction index of earth, a the wire radius, d the spacing between the
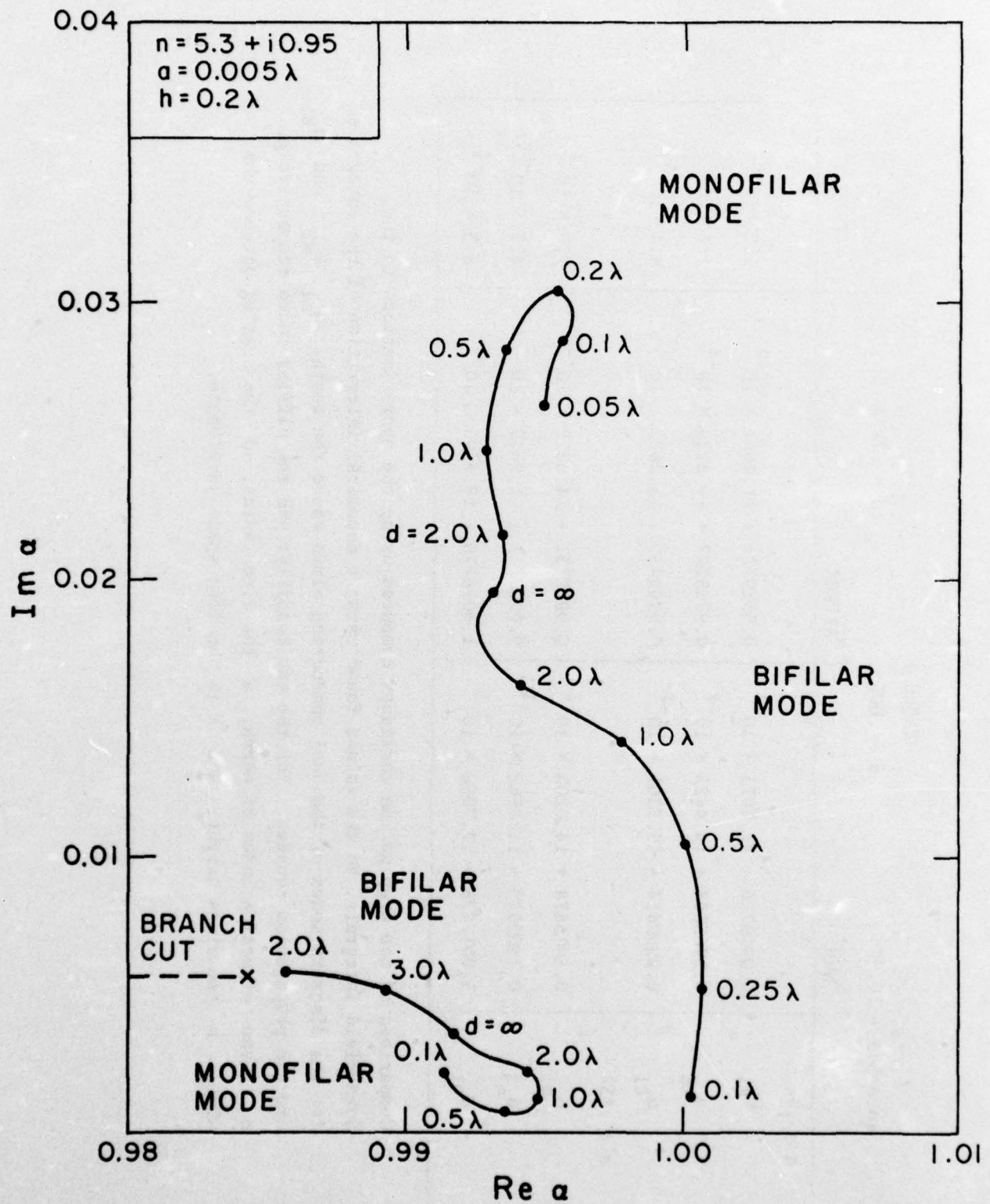wires, h the wires height, and λ is the free space wavelength.

Fig. 2: Mode trajectories as a function of wire spacing (h = 0.2λ)

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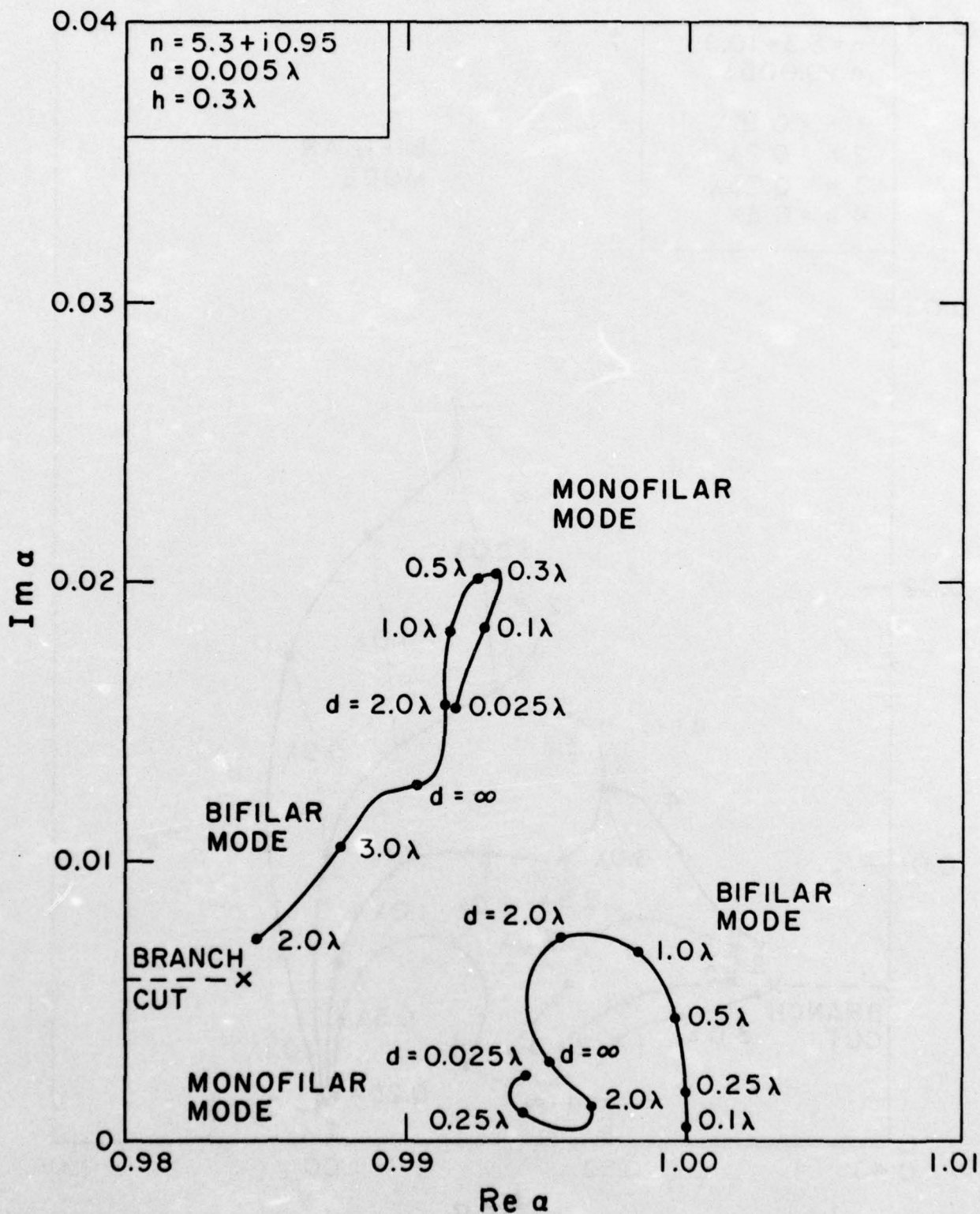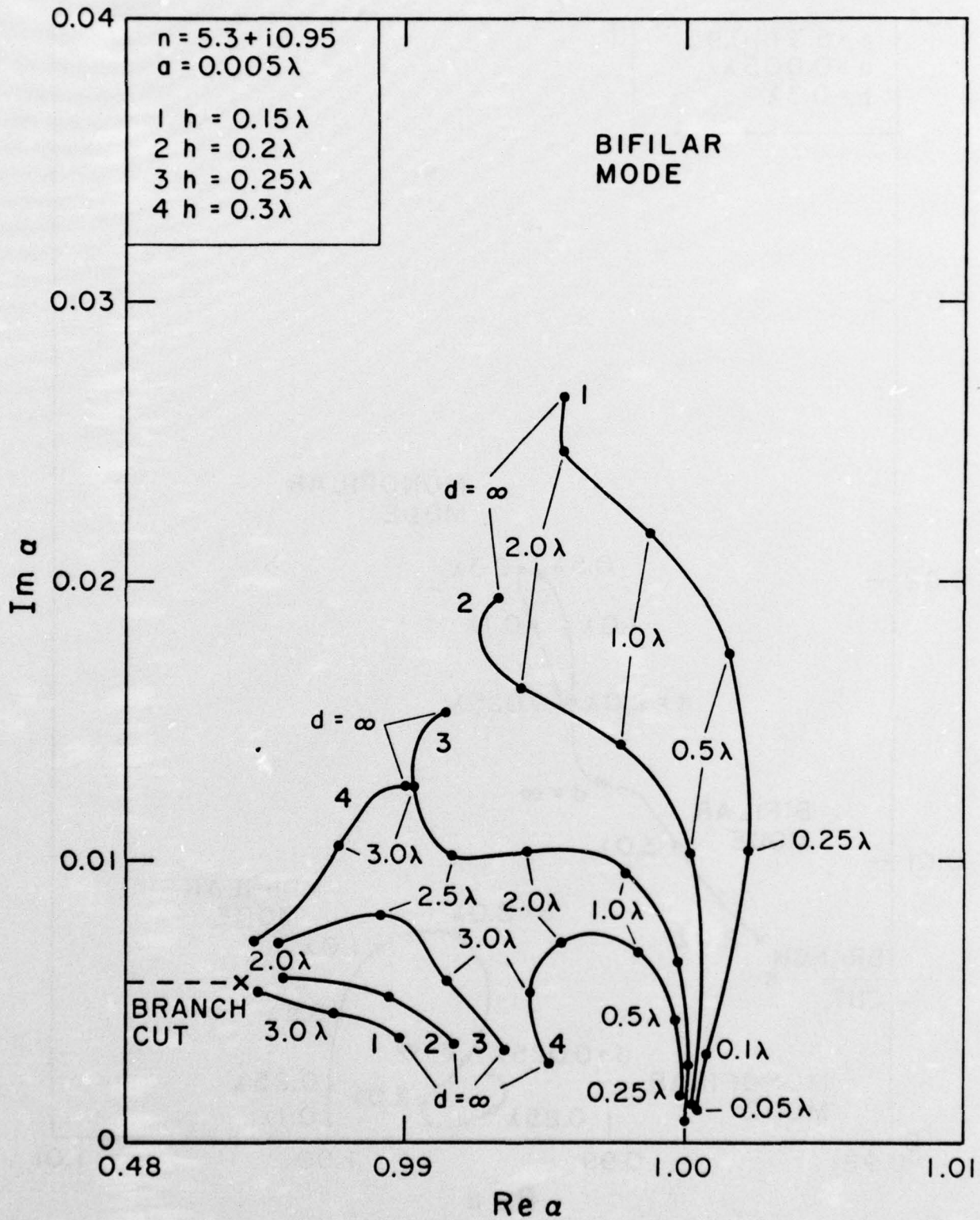Fig. 3: Mode trajectories as a function of wire spacing (h = 0.3λ)

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Fig. 4: Mode trajectories as a function of wire spacing for various wire heights
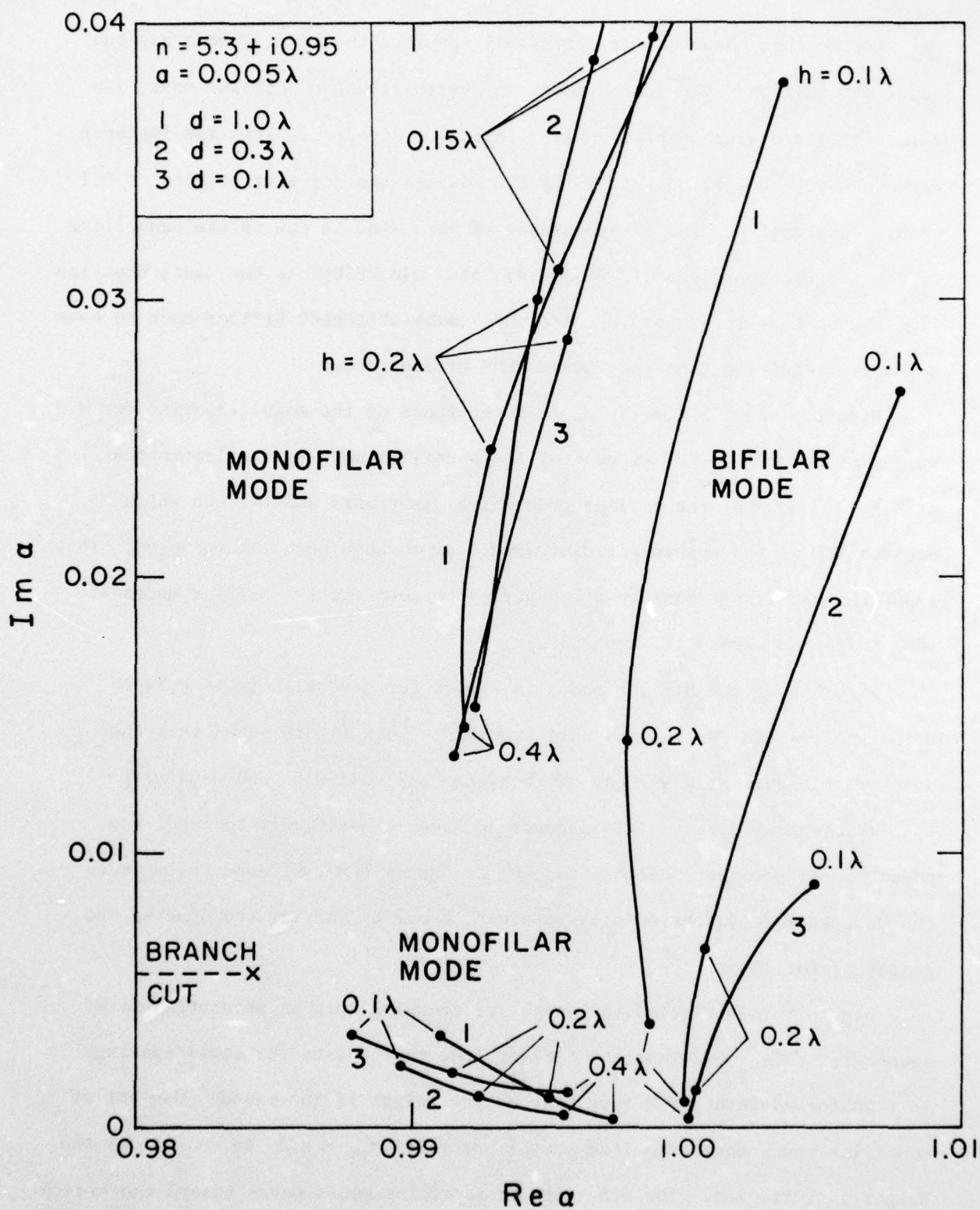
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Fig. 5: Mode trajectories as a function of wire height for various wire spacings

the wire height; however for very small spacings the bifilar mode has the lower attenuation. The fourth mode, an "earth-attached" bifilar mode, can exist for large wire spacings only. This mode disappears into the improper Riemann sheet at spacings of 1.5 to 2.0 wavelengths for wire heights of 0.1 to 0.4 wavelengths. The disappearance of this mode is due to the cancelling of the singular portions of $Q(\alpha;2H,0)$ and $Q(\alpha;2H,D)$ in the modal equation (12). Note that it is possible for the "earth-attached" bifilar mode to have a lower attenuation than the "quasi TEM" bifilar mode.

Figure 3 shows a similar plot of the roots of the modal equation for a height of $h = 0.3\lambda$. The movement of the monofilar mode is similar to that of Fig. 2; however the bifilar mode which disappears and the one which becomes TEM as the spacing is decreased seem to have been interchanged. This indicates that there must be a degeneracy between the two bifilar modes at some height between 0.15λ and 0.3λ .

Figure 4 of the bifilar modes is a plot for several heights between 0.15λ and 0.3λ and for various wire spacings. This figure shows that the degeneracy occurs at a wire height between 0.25λ and 0.3λ and at a wire spacing of about 2.5λ . This degeneracy makes it difficult to label the modes as being either "earth-attached" or "quasi TEM", because these modes can be transformed continuously into each other by varying the spacing and height of the wires.

Figure 5 is a plot of the modes for changing heights at several fixed spacings, This shows that the bifilar mode that exists for small spacings is transformed into a TEM type mode as the height is increased. One set of monofilar modes moves from the branch point at $\alpha_{BP} = n/\hat{n}$ to $\alpha = 1$ as the height is increased. The other set of monofilar modes moves toward the branch point as the height is increased.

VI. CONCLUSION

In this report, we have investigated the modes of propagation along a two-wire line parallel to and above the surface of a finitely conducting earth. Due to interaction between the two wires as well as between the wires and the earth, the mode structure is more complex than that which would be found in the case of a perfectly conducting ground - a single monofilar and a single bifilar mode. Moreover, the existence of modal degeneration, similar to that discovered in [1] for the single-wire line, has been demonstrated.

Systematic analytic approximations have been derived for the Sommerfeld integrals which enter into the modal equation, as well as rigorous error bounds for them. These expressions allow a great savings in the computer time required to numerically determine the modal propagation constants.

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APPENDIX A

In this Appendix the higher order terms of P are derived and error bounds for these are found. An (M+1)th order approximation is obtained by extracting the first M+1 terms in the expansion of u_2 in (18) from u_2 in (17). P can then be written as

$$P(\alpha;X;Y) = \sum_{k=0}^{M} P_k(\alpha;X;Y) + \varepsilon_{PM} \tag{A.1}$$

where P_0 is given in (20), and for k>0

$$P_k(\alpha;X;Y) = \frac{2i\zeta_n}{N^2 k!} \zeta_n^{-2k} C_k I_k(\alpha;X;Y) \tag{A.2}$$

and where

$$C_k = (\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2}) \ldots (\tfrac{3}{2} - k) \tag{A.3}$$

and

$$
\begin{aligned}
I_k(\alpha;X;Y) &= (i\pi)^{-1} \int_{-\infty}^{\infty} \lambda^{2k} \exp(-u_1 X - i\lambda Y)d\lambda \\
&= (i\pi)^{-1}\{[-\partial^2/\partial Y^2]^k [-\partial/\partial X] \int_{\infty}^{\infty} [\exp(-u_1 X - i\lambda Y)/u_1]d\lambda\} \\
&= \{[-\partial^2/\partial Y^2]^k \zeta X(X^2 + Y^2)^{-\frac{1}{2}} H_1^{(1)}[\zeta(X^2 + Y^2)^{\frac{1}{2}}]\}
\end{aligned}
\tag{A.4}
$$

with the use of the identity given in (21). In particular, for the 2-terms expansion

$$P_1(\alpha;X;Y) = (i/N^2\zeta_n)\{(\zeta X/R^5)H_1^{(1)}(\zeta R) [6Y^2 - 2X^2 - \zeta^2 Y^2 R^2] + (\zeta^2 X^2/R^4)H_o^{(1)}(\zeta R)[X^2 - 3Y^2]\} \tag{A.5}$$

Combining (A.5) and (19) we obtain the following expression for $P(\alpha;X;Y)$:

$$
\begin{aligned}
P(\alpha;X;Y) = \frac{2}{N^2} \{ & \frac{\zeta}{R^3} H_1^{(1)}(\zeta R)[X^2 - Y^2 + i\zeta_n XR^2 + \frac{iX}{2\zeta_n R^2}(6Y^2 - 2X^2 - \zeta^2 Y^2 R^2)] \\
& - \frac{\zeta^2 X}{R^2} H_o^{(1)}(\zeta R) [X + \frac{i}{2\zeta_n R^2}(3Y^2 - X^2)]\}
\end{aligned}
\tag{A.6}
$$

An error bound to the first order approximation to $P(\alpha;X;Y)$ can be obtained as follows:

$$|\varepsilon_{PO}| = \left|-\frac{2}{i\pi N^2} \int_{-\infty}^{\infty} (-u_2-i\zeta_n)\exp(-u_1 X - i\lambda Y)d\lambda\right|$$

$$\leq \frac{4}{\pi|N^2|} \int_{0}^{\infty} \frac{\lambda^2}{|u_2-i\zeta_n|} \exp[-X\,\mathrm{Re}(u_1)]d\lambda \tag{A.7}$$

Let

$$\delta = \begin{cases} 0 & [\mathrm{Re}(\zeta^2) \leq 0], \\[2mm] (\mathrm{Re}\,\zeta^2)^{\frac{1}{2}} & [\mathrm{Re}(\zeta^2) > 0], \end{cases} \tag{A.8}$$

then it can be shown that

$$\mathrm{Re}(u_1) = \mathrm{Re}(\lambda^2-\zeta^2)^{\frac{1}{2}} \geq \begin{cases} 0 & (\lambda < \delta), \\[2mm] (\lambda^2-\delta^2)^{\frac{1}{2}} & (\lambda \geq \delta). \end{cases}$$

Now $(\lambda^2-\delta^2)^{\frac{1}{2}} \geq \lambda - \delta$, if $\lambda \geq \delta$. **So it follows that**

$$\mathrm{Re}(u_1) \geq \begin{cases} 0 & (\lambda < \delta), \\[2mm] \lambda-\delta & (\lambda > \delta). \end{cases} \tag{A.9}$$

Now u_2 and $-i\zeta_n$ are in the same quadrant for all real values of λ, so $|u_2-i\zeta_n| \geq |\zeta_n|$. Using these relations, (A.7) becomes

$$|\varepsilon_{PO}| \leq \frac{4}{\pi|N^2\zeta_n|} \left\{ \int_{0}^{\delta} \lambda^2 d\lambda + \int_{\delta}^{\infty} \lambda^2 \exp[-X(\lambda-\delta)]d\lambda \right\}$$

$$= \frac{4[2+2\delta X + \delta^2 X^2 + \delta^3 X^3/3]}{|\pi N^2\zeta_n|X^3} \tag{A.10}$$

Similarly the error in the second order approximation to $P(\alpha;X;Y)$ is

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$$|\varepsilon_{P1}| = |\frac{4}{i\pi N^2} \int_0^\infty (-u_2 - i\zeta_n + \frac{i\lambda^2}{2\zeta_n}) \exp(-u_1 X - i\lambda Y) d\lambda|$$

$$\leq \frac{4}{\pi|N^2|} \int_0^\infty \frac{\lambda^4}{|2\zeta_n(u_2 - i\zeta_n)^2|} \exp(-X\text{Re}\,u_1) d\lambda$$

$$\leq \frac{2}{\pi|N^2\zeta_n^3|} \{ \int_0^\delta \lambda^4 d\lambda + \int_0^\delta \lambda^4 \exp[-X(\lambda-\delta)] d\lambda \}$$

$$= \frac{2[24 + 24\delta X + 12\delta^2 X^2 + 4\delta^3 X^3 + \delta^4 X^4 + \delta^5 X^5/5]}{\pi|N^2\zeta_n^3|X^5} \tag{A.11}$$

By comparing (A.10) with (A.11), one can see that the error in P is decreased by adding the second term if $|n^2|X^2 > 6$.

APPENDIX B

In this Appendix an expression for $W_o(\alpha;Y)$ is derived. $W_o(\alpha;Y)$ as given in (26) can be manipulated as follows

$$W_o(\alpha;Y) = \int_{-\infty}^{\infty} \frac{\exp(-i\lambda Y)d\lambda}{u_1(u_1-i/\hat{n})}$$

$$= \int_{-\infty}^{\infty} \frac{\exp(-i\lambda Y)d\lambda}{\lambda^2 - \lambda_p^2} + i/\hat{n} \int_{-\infty}^{\infty} \frac{\exp(-i\lambda Y)d\lambda}{u_1(\lambda^2-\lambda_p^2)}$$

$$= W_{01}(\alpha;Y) + i/\hat{n}\, W_{02}(\alpha;Y) \qquad (B.1)$$

$W_{01}(\alpha;Y)$ is found by deforming the contour into the lower half plane and evaluating the residue to obtain

$$W_{01}(\alpha;Y) = \frac{\pi i}{\lambda_p}\, \exp(i\lambda_p Y) \qquad (B.2)$$

where $0 \leq \arg \lambda_p < \pi$. On the other hand we can rearrange W_{02} to obtain

$$W_{02}(\alpha;Y) = \frac{1}{2\lambda_p} \left\{ \int_{-\infty}^{\infty} \frac{\exp(-i\lambda Y)d\lambda}{u_1(\lambda-\lambda_p)} - \int_{-\infty}^{\infty} \frac{\exp(-i\lambda Y)d\lambda}{u_1(\lambda+\lambda_p)} \right\}$$

$$= \frac{1}{2\lambda_p} \left\{ I_1(\alpha;Y) - I_2(\alpha;Y) \right\} \qquad (B.3)$$

The integral I_1 can be written as

$$I_1(\alpha;Y) = \exp(-i\lambda_p Y)\left\{ \int_{-\infty}^{\infty} \frac{(\exp[-iY(\lambda-\lambda_p)]-1)}{u_1(\lambda-\lambda_p)}d\lambda + \int_{-\infty}^{\infty} \frac{d\lambda}{u_1(\lambda-\lambda_p)} \cdot \right\}$$

$$= \exp(-i\lambda_p Y)\left\{ -i \int_{0}^{Y}\exp(is\lambda_p) \int_{-\infty}^{\infty} \frac{\exp(-is\lambda)}{u_1}d\lambda ds + \lambda_p \int_{-\infty}^{\infty} \frac{d\lambda}{u_1(\lambda^2-\lambda_p^2)} \right\}$$

$$I_1(\alpha,Y) = \exp(-i\lambda_p Y)\left\{\pi \int_0^Y \exp(is\lambda_p)H_o^{(1)}(\zeta s)ds + \lambda_p \int_{-\infty}^\infty \frac{d\lambda}{u_1(\lambda^2-\lambda_p^2)}\right\} \qquad (B.4)$$

Similarly I_2 can be written as

$$I_2(\alpha,Y) = \exp(i\lambda_p Y)\left\{\pi \int_0^Y \exp(-is\lambda_p)H_o^{(1)}(\zeta s)ds - \lambda_p \int_{-\infty}^\infty \frac{d\lambda}{u_1(\lambda^2-\lambda_p^2)}\right\} \qquad (B.5)$$

By replacing these expressions for I_1 and I_2 into (B.3), we obtain

$$W_{02}(\alpha;Y) = (\pi/i\lambda_p) \int_0^Y \sin[\lambda_p(Y-s)]H_o^{(1)}(\zeta s)ds$$

$$+ \cos(\lambda_p Y)W_{02}(\alpha;0) \qquad (B.6)$$

where

$$W_{02}(\alpha;0) = \int_{-\infty}^\infty \frac{d\lambda}{u_1(\lambda^2-\lambda_p^2)}$$

An expression for $W_{02}(\alpha,0)$ was found by Olsen and Chang [20] in which the contour is deformed around the branch cut, and to this integration was added the residue of the pole at λ_p. The resulting expression is

$$W_{02}(\alpha;0) = (\hat{n}/i\lambda_p)\{2[\ln(\zeta)-\ln(1/\hat{n}-i\lambda_p)] - i\pi\} \qquad (B.7)$$

The principal value of the logarithms are chosen. The substitution of (B.2), (B.6), and (B.7) into (B.1) then yields

$$W_o(\alpha;y) = (2/\lambda_p)\cos(\lambda_p Y)[\ln(\zeta) - \ln(1/\hat{n} - i\lambda_p)] - (\pi/\lambda_p)\sin(\lambda_p Y)$$

$$+ W_3(\alpha,Y) \qquad (B.8)$$

where

$$W_3(\alpha;Y) = (\pi/\lambda_p\hat{n}) \int_0^Y \sin[\lambda_p(Y-s)]H_o^{(1)}(\zeta s)ds \qquad (B.9)$$

This can also be written as

$$W_3(\alpha;Y) = (\pi i/2\lambda_p \hat{n})\{\exp(-i\lambda_p Y)W_4(\zeta,Y,i\lambda_p)$$

$$- \exp(i\lambda_p Y)W_4(\zeta,Y,-i\lambda_p)\} \qquad (B.10)$$

where

$$W_4(\zeta,Y,t) = \int_0^Y \exp(ts)H_0(\zeta s)ds \qquad (B.11)$$

This integral can be evaluated by expanding the exponential into a power

series and integrating the series term by term. The resulting expression

is

$$W_4(\zeta,Y,t) = \sum_{m=0}^{\infty} \frac{(tY)^m Y}{m!} I_m(\zeta Y) \qquad (B.12)$$

where

$$I_m(\zeta Y) = \int_0^1 s^m H_0^{(1)}(\zeta Y s)ds \qquad (B.13)$$

These integrals can be found from the recursion relation for $m \geq 2$

$$I_m(\zeta Y) = (\zeta Y)^{-1}H_1^{(1)}(\zeta Y) + (m-1)(\zeta Y)^{-2}H_0^{(1)}(\zeta Y) - (m-1)^2(\zeta Y)^{-2}I_{m-2}(\zeta Y)$$

$$(B.14)$$

I_0 and I_1 can be expressed in closed form as

$$I_0(\zeta Y) = H_0^{(1)}(\zeta Y) + (\pi/2)[S_0(\zeta Y)H_1^{(1)}(\zeta Y) - S_1(\zeta Y)H_0^{(1)}(\zeta Y)] \qquad (B.15)$$

Here $S_j(x)$ is the Struve function of order j, and

$$I_1(\zeta Y) = (\zeta Y)^{-1}[H_1^{(1)}(\zeta Y) + 2i/(\pi \zeta Y)] \qquad (B.16)$$

These results were obtained from Olsen and Chang [20].

APPENDIX C

In this Appendix an expression for $W_x(\alpha;X;Y)$ is derived. $W_x(\alpha;X;Y)$ as given in (26) can be written as

$$W_x(\alpha;X;Y) = -\int_0^X \exp(is/\hat{n}) \int_{-\infty}^{\infty} [\exp(-su_1 - i\lambda Y)/u_1] \, d\lambda ds$$

$$= -i\pi \int_0^X \exp(is/\hat{n}) H_o^{(1)} [\zeta(s^2+Y^2)^{\frac{1}{2}}] ds \qquad (C.1)$$

In the case that $Y = 0$, this can be expressed as

$$W_x(\alpha;X;0) = -i\pi W_4(\zeta,X,1/\hat{n})$$

where an expansion for W_4 is given in Appendix B, in equation (B.12) through (B.16).

An expansion for W_x is the region Y large compared to X is found as follows. Consider the function $f(\theta)$ defined by $f(\theta) = H_o^{(1)}(\zeta\theta^{\frac{1}{2}})$. This can be expanded into a Taylor series around θ_c as

$$f(\theta) = \sum_{m=0}^{\infty} \frac{f^{[m]}(\theta_c)}{m!} (\theta-\theta_c)^m \qquad (C.2)$$

The derivatives of f with respect to θ, are given by the recursion formula for $m \geq 2$

$$f^{[m]}(\theta) = -\theta^{-1}[(m-1)f^{[m-1]}(\theta) + (\zeta^2/4)f^{[m-2]}(\theta)]$$

where

$$f'(\theta) = (-\zeta/2\theta^{\frac{1}{2}})H_1^{(1)}(\zeta\theta^{\frac{1}{2}})$$

Now, the substitution of $\theta = s^2 + Y^2$ and $\theta_c = X^2/2 + Y^2$ yields

$$H_o^{(1)} [\zeta(s^2+Y^2)^{\frac{1}{2}}] = \sum_{m=0}^{\infty} \frac{f^{[m]}(X^2/2+Y^2)}{m!} (s^2-X^2/2) \qquad (C.3)$$

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By inserting (C.3) into (C.1) we have

$$W_x(\alpha; X, Y) = -i\pi \sum_{m=0}^{\infty} \frac{f^{|m|}(X^2/2 + Y^2)}{m!} \, x^{2m+1} \, I_m(iX/\hat{n}) \tag{C.4}$$

where

$$I_m(t) = \int_0^1 (s^2 - 1/2)^m \exp(ts) ds \tag{C.5}$$

The integrals $I_m(t)$ can be expressed in terms of known functions through the use of the recursion formula for $m \geq 2$

$$I_m(t) = t^{-1}\{2^{-m}[\exp(t) - (-1)^m] - (2m/t)[2^{1-m}\exp(t)]$$

$$- (2m-1)I_{m-1}(t) - (m-1)I_{m-2}(t)]\} \tag{C.6}$$

with

$$I_0(t) = [\exp(t) - 1]/t$$

$$I_1(t) = \{\exp(t) - (2/t)[\exp(t) - I_0(t)]\}/t - I_0(t)/2$$

However for large m and small t this method leads to a large amount of roundoff error because of the large number of cancellations. A better method for calculating I_m for large m and small t is to expand the exponential into a Taylor series around zero to obtain

$$I_m(t) = \sum_{k=1}^{\infty} \frac{(t)^k}{k!} \, K_{m,k} \tag{C.7}$$

where

$$K_{m,k} = \int_0^1 (s^2 - 1/2)^m s^k ds$$

which can be computed from the recursion formula for $m \geq 1$

$$K_{m,k} = \frac{2^{-m} - mK_{m-1,k}}{k + 2m + 1} \tag{C.8}$$

with

$$K_{o,k} = 1/(k+1)$$

We note that $I_m(t)$ is not a function of α or Y, so these can be computed once for finding several roots with constant n and X.

We also note that the series for W_x given in (C.4) converges similarly to the series

$$\sum_{m=0}^{\infty} \left(\frac{X^2}{X^2 + 2Y^2} \right)^m$$

This series converges for all $Y > 0$, even though the rate of convergence decreases for small Y.

APPENDIX D

In this Appendix an expression for $W(\alpha;X;Y)$ is found that is good in the region Y smaller than X. The cosine term in (31) is expanded into a power series of Y, and this series is integrated term by term to obtain

$$W(\alpha;X;Y) = \sum_{m=0}^{\infty} \frac{(-1)^m Y^{2m}}{(2m)!} I_m(\alpha;X) \tag{D.1}$$

where

$$I_m(\alpha;X) = \int_{-\infty}^{\infty} \frac{\lambda^{2m}\exp(-u_1 X)}{u_1(u_1-i/\hat{n})} \, d\lambda$$

$$= [\partial^2/\partial X^2 + \zeta^2]^m \, W(\alpha;X,0)$$

where

$$W(\alpha;X,0) = \int_{\infty}^{\infty} \frac{\exp(-u_1 X)}{u_1(u_1-i/\hat{n})} \, d\lambda$$

An expression for $W(\alpha;X,0)$ is found by setting $Y = 0$ in (27) and (29), and inserting these into (26) to obtain

$$W(\alpha;X,0) = \exp(-iX/\hat{n})\{-i\pi \int_0^X \exp(is/\hat{n})H_o^{(1)}(\zeta s)ds$$

$$+ (2/\lambda_p)[\ell n(\zeta) - \ell n(1/\hat{n}-i\lambda_p)]\} \tag{D.2}$$

A series expansion for the finite integral above is given in Appendix B, in equations (B.11) through (B.16). Let I_m be written as

$$I_m(\alpha;X) = \lambda_p^{2m}W(\alpha;X,0) - i\pi R_m(\alpha;X) \tag{D.3}$$

A recursion relation for R_m is found to be for $m \geq 1$

$$R_m(\alpha;X) = (\lambda_p^2 + \zeta^2)R_{m-1}(\alpha;X) + f_{m-1}(\alpha;X) \qquad (D.4)$$

where

$$f_m(\alpha;X) = [\partial^{2m}/\partial X^{2m}][\partial/\partial X - i/\hat{n}]H_o^{(1)}(\zeta X)$$

and

$$R_o(\alpha;X) = 0$$

Inserting (D.3) into (D.1) one obtains

$$W(\alpha;X;Y) = \cos(\lambda_p Y)W(\alpha;X,0) - i\pi \sum_{m=1}^{\infty} \frac{(-1)^m Y^{2m}}{(2m)!} R_m(\alpha;X) \qquad (D.5)$$

We note that for small ζX the function f_m is asymptotic to $(-1)^m(2m)!/(X^{2m})H_1^{(1)}(\zeta X)$. This implies that the series in (D.5) converges similarly to the series

$$\sum_{m=1}^{\infty} \frac{(Y/X)^{2m}}{(2m)(2m-1)}$$

This series converges if and only if $Y < X$, so we can infer that the series in (D.5) converges in the region $Y < X$.

APPENDIX E

In this Appendix higher order terms in the approximation of Q will be derived and error bounds for the first few terms will be found. As with P an (M+1)th order approximation to Q is found by extracting the first M+1 terms in the expansion of u_2 in (24) from u_2 in (23). Q is then written as

$$Q(\alpha;X,Y) = \sum_{k=0}^{M} Q_k(\alpha;X,Y) + \varepsilon_{QM} \tag{E.1}$$

where Q_k is the portion due to the kth term in the expansion of u_2. Q_o is given in (25), and for $k > 0$

$$Q_k(\alpha;X,Y) = \frac{-2i\alpha^2 n^2 C_k}{(n^4-1)\hat{n}k!} \left(\frac{\hat{n}}{n^2}\right)^{2k} I_k(\alpha;X,Y) \tag{E.2}$$

where

$$C_k = (\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2}) \ldots (\tfrac{3}{2} - k)$$

and

$$I_k(\alpha;X,Y) = (i\pi)^{-1} \int_{-\infty}^{\infty} (\lambda^2 - \lambda_p^2)^{k-1} \exp(-u_1 X - i\lambda Y) d\lambda$$

$$= (i\pi)^{-1}\{[\partial^2/\partial X^2 + \hat{n}^{-2}]^{k-1}[-\partial/\partial X] \int_{-\infty}^{\infty} [\exp(-u_1 X - i\lambda Y)/u_1] d\lambda\}$$

$$= \{[\partial^2/\partial X^2 + \hat{n}^{-2}]^{k-1} \zeta X (X^2+Y^2)^{-\frac{1}{2}} H_1^{(1)} [\zeta(X^2+Y^2)^{\frac{1}{2}}]\}$$

In particular, Q_1 is found to be

$$Q_1(\alpha;X,Y) = \frac{-i\alpha^2 \hat{n} \zeta X}{(n^4-1)n^2 R} H_1^{(1)}(\zeta R) \tag{E.3}$$

The error in approximating Q by Q_o is given by

$$\left|\epsilon_{QO}\right| = \frac{2}{\pi}\left|\frac{\alpha^2}{(n^4-1)} \int_{-\infty}^{\infty} \frac{-in^2/\hat{n}-u_2}{\lambda^2-\lambda_p^2} \exp(-u_1 X-i\lambda Y)d\lambda\right|$$

$$\leq \frac{2}{\pi}\left|\frac{\alpha^2}{(n^4-1)}\right| \int_{-\infty}^{\infty} \frac{\exp[-X\operatorname{Re}(u_1)]}{\left|u_2-in^2/\hat{n}\right|} d\lambda \qquad (E.4)$$

If $I_m(\alpha^2) \leq I_m(n^2)$, then both u_2 and $-in^2/\hat{n}$ are in the fourth quadrant in the complex plane. This implies that $\left|u_2-in^2/\hat{n}\right| \geq \left|n^2/\hat{n}\right|$. Using the bounds on $\operatorname{Re}(u_1)$ given in Appendix A, the error bound on Q_o becomes

$$\left|\epsilon_{QO}\right| \leq \frac{4}{\pi}\left|\frac{\alpha^2\hat{n}}{(n^4-1)n^2}\right| \frac{(1+\delta X)}{X} \qquad (E.5)$$

where

$$\delta = \begin{cases} (\operatorname{Re}\zeta^2)^{\frac{1}{2}} & (\operatorname{Re}\zeta^2 > 0), \\ 0 & (\operatorname{Re}\zeta^2 \leq 0). \end{cases}$$

Similarly the error in approximating Q by Q_o+Q_1 is bounded by

$$\left|\epsilon_{Q_1}\right| \leq \frac{2}{\pi}\left|\frac{\alpha^2\hat{n}}{(n^4-1)n^2}\right| \int_{\infty}^{\infty} \frac{\left|\lambda^2-\lambda_p^2\right|}{\left|u_2-in^2/n\right|^2} \exp[-X\operatorname{Re}(u_1)]d\lambda \qquad (E.6)$$

Using the same inequalities used in obtaining (E.5), then (E.6) reduces to

$$\left|\epsilon_{Q_1}\right| \leq \frac{2}{\pi}\left|\frac{\alpha^2\hat{n}^3}{(n^4-1)n^6}\right| \frac{[2+2\delta X+\delta^2 X^2+\delta^3 X^3/3 +\left|\lambda_p^2\right|X^2(\delta X+1)]}{X^3} \qquad (E.7)$$

By comparing (E.7) with (E.5) it is evident that the error is decreased by including Q_1 if $\left|n^2\right|X^2 > 1$.

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